

The problem of characterising stationary data for the vacuum Einstein equations

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Abstract

We take a step towards characterising stationary data for the vacuum Einstein equations, by finding a necessary condition on initial data for which the evolution is a solution of the vacuum equations admitting a Killing vector, which is time-like at least in some region of the Cauchy development.

1 Introduction: the Problem

In this article we consider the problem of characterising stationary data for the vacuum Einstein equations, that is to say of identifying data for which the evolution is a solution of the vacuum equations admitting a Killing vector which is time-like at least in some region of the Cauchy development. By data here we mean a 3-manifold Σ with a Riemannian metric g_{ij} and a symmetric tensor K_{ij} , understood as the second fundamental form of Σ , together satisfying the vacuum constraint equations (1)–(2) below. We obtain a necessary condition expressed in terms of the data, and a simple interpretation of it in terms of the associated space-time.

A particular case of this problem, when $K_{ij} = 0$, was considered in [1]. Necessary conditions were found for staticity, and the problem was completely solved in the case when the Ricci tensor of g_{ij} has distinct eigenvalues¹. We now consider the general case when $K_{ij} \neq 0$. Note that the

¹If this Ricci tensor has a repeated eigenvalue then the space-time is type D, and all

method detects the presence of space-time Killing vectors and it is then a secondary question whether the Killing vector is time-like in a suitable region. Thus we shall be addressing the larger question of data leading to space-time symmetries.

We consider data subject to the vacuum constraint equations:

$$R + K^2 - K_{ij}K^{ij} = 0, \quad (1)$$

$$D_i K^i_j - D_j K = 0, \quad (2)$$

where D_i is the Levi-Civita covariant derivative, R the Ricci scalar of Σ and $K = g^{ij}K_{ij}$.

Beig and Chruściel [2] introduced the notion of Killing Initial Data or KIDs as follows: if the space-time M which evolves from this data admits a Killing vector X then X can be decomposed with respect to the normal to Σ as a pair (N, Y^i) consisting of a function N on Σ and a vector field Y^i tangent to Σ and, by virtue of the Killing equations in M , this pair satisfies the pair of equations:

$$D_{(i}Y_{j)} = -NK_{ij} \quad (3)$$

$$D_i D_j N + \mathcal{L}_Y K_{ij} = N(R_{ij} + K K_{ij} - 2K_{im}K_j^m). \quad (4)$$

Further, given the data and the KID (N, Y^i) , the same authors introduce the *Killing development*, [2], which is the 4-metric:

$$g = g_{ij}(dx^i + Y^i du)(dx^j + Y^j du) - N^2 du^2, \quad (5)$$

and is a solution of the vacuum field equations admitting a Killing vector $X = \partial/\partial u$ and inducing the original data at $\Sigma = \{u = 0\}$. The Killing vector is evidently time-like, space-like or null according to the sign of $N^2 - g_{ij}Y^i Y^j$.

Beig et al [3] show that KIDs are non-generic, in the sense that generic sets of data do not admit them, they prove the existence of data with none, and they give some necessary conditions on data for non-existence of a KID.

Our interest here is in the conditions for non-existence, and we shall obtain sufficient conditions which are simpler than those in [3], and with a direct geometric interpretation. The results are contained in the following proposition:

Proposition 1.1 *1. There can be no KIDs if the determinant of a certain 10×10 -matrix A is nonzero. The entries of A are polynomials in the tensors $R_{ij}, D_i R_{jk}, K_{ij}, D_i K_{jk}$ and $D_i D_j K_{km}$ where R_{ij} is the Ricci tensor of g_{ij} .*

static vacuum type D solutions are known explicitly, [6], so the problem is also solved in this case.

2. *Using 3 + 1-formalism, the entries of A can be expressed directly in terms of the Weyl tensor of the space-time evolving from the data, and the determinant condition in part 1 can then be written*

$$dI \wedge dJ \wedge d\bar{I} \wedge d\bar{J} \neq 0. \quad (6)$$

where I and J are the two complex scalar invariants of the Weyl spinor defined in (20) below.

If the determinant is zero, then there will be candidate KIDs, and this will always be the case for example with data for algebraically special space-times when necessarily $I^3 - 6J^2$ vanishes. To see whether the candidate KIDs are actually KIDs then requires derivatives of the constraints given by A , and we won't pursue this question here.

The method of proof is to prolong the system (3)–(4) and obtain a first-order linear system for a larger set of variables (see [7] and [8] for earlier applications of the method). This linear system can be regarded as covariant constancy for a section of a particular vector bundle with connection, and existence of solutions can then be analysed in terms of the curvature of the connection. In Section 2 we find a sufficient condition for non-existence of a KID, equivalently a condition that the data not be data for a stationary space-time², in terms of the non-vanishing of a determinant expressed in terms of the data and its derivatives. In Section 3, we express this condition in space-time terms, as in (6), when it is more perspicuous.

We shall adopt the convention that indices from the start of the alphabet, a, b, c, \dots are 4-dimensional indices, that is indices on tensors in the space-time M , and indices from the middle of the alphabet, i, j, k, \dots are 3-dimensional indices, on tensors tangent to Σ . With n_a as the unit time-like normal to Σ in M we therefore have

$$N = X^a n_a, \quad Y^i = \Pi_a^i X^a,$$

where Π_a^i is the projection into Σ . To simplify equations, but with a slight abuse of notation, we shall usually omit the projection tensors Π_a^i and Π_i^a .

²Or indeed a space-time admitting any Killing vector.

2 The bundle and connection

We begin by prolonging (3)–(4). Introduce new variables $V_i := D_i N$ and $M_{ij} := D_{[i} Y_{j]}$ so that

$$D_i N = V_i \quad (7)$$

$$D_i Y_j = -N K_{ij} + M_{ij} \quad (8)$$

$$D_i V_j = N(R_{ij} + K K_{ij}) - Y^k D_k K_{ij} - K_{im} M_j^m - K_{jm} M_i^m. \quad (9)$$

We need an equation for $D_i M_{jk}$. Commute derivatives on Y_j and rearrange to find

$$D_k M_{ij} = -2V_{[i} K_{j]k} - 2N D_{[i} K_{j]k} - R_{ijk\ell} Y^\ell,$$

with the convention

$$(D_i D_j - D_j D_i) Z_k = R_{ijk}{}^\ell Z_\ell.$$

Then by relabelling and reordering

$$D_i M_{jk} = -2N D_{[j} K_{k]i} - R_{jki\ell} Y^\ell - 2V_{[j} K_{k]i}, \quad (10)$$

and we have the desired linear system (7)–(10).

To see this system as a connection on a vector bundle, introduce the ten component column-vector $\Psi_\alpha = (N, Y_j, V_j, M_{jk})^T$, which is a section of the vector bundle $\Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma) \oplus \Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma)$, when the system appears as

$$\mathcal{D}_i \Psi_\alpha := D_i \Psi_\alpha - \Gamma_{i\alpha}{}^\beta \Psi_\beta = 0, \quad (11)$$

with $\Psi_\beta = (N, Y_p, V_p, M_{pq})^T$ and

$$\Gamma_{i\alpha}{}^\beta = \begin{pmatrix} 0 & 0 & \delta_i^p & 0 \\ -K_{ij} & 0 & 0 & \delta_i^p \delta_j^q \\ R_{ij} + K K_{ij} & -D^p K_{ij} & 0 & -K_i^p \delta_j^q - K_i^q \delta_j^p \\ -2D_{[j} K_{k]i} & -R_{jki}{}^p & -2\delta_{[j}^p K_{k]i} & 0 \end{pmatrix}. \quad (12)$$

Note that Ψ contains all the components of X^a , namely N and Y^i , and all the components of $\nabla_a X_b$ since, as one may calculate,

$$\nabla_a X_b = M_{ab} + \tilde{V}_a n_b - n_a \tilde{V}_b, \quad (13)$$

where

$$\tilde{V}_a = V_a + K_{ab} Y^b,$$

so that V_i and M_{ij} with Y^i and the data determine all components of the derivative of X .

We won't need it but for completeness note

$$X^a = Nn^a + Y^a,$$

with

$$n^a N_{,a} = -Y^a A_a,$$

and

$$P_a^d n^c \nabla_c Y_d = \tilde{V}_a - N A_a$$

where P_a^b is projection orthogonal to n^a and A^a is the acceleration of the normal congruence i.e.

$$\nabla_a n_b = -n_a A_b + K_{ab}.$$

The acceleration is typically fixed by a gauge condition but, for our purposes, we don't need to do that.

The usual argument can now be followed: a solution of the prolonged system corresponds to a constant section of the vector bundle in the connection \mathcal{D} and so must annihilate the curvature tensor of \mathcal{D} ; in particular the rank of this curvature must be less than maximal, and this will be the source of the desired conditions on the data.

Rather than writing out the curvature tensor, we note that commutators on N and V_j have been seen to give identities. New conditions arise only from the commutator of Levi-Civita derivatives on V_j and M_{jk} and for these we find:

- By considering $\epsilon^{ij}_m D_i D_j V_n$ and using (7)–(10) we deduce the vanishing of

$$\begin{aligned} C_{mn}^{(1)} = & N \epsilon^{ij}_m (D_i R_{jn} + K_{,i} K_{jn} + K D_i K_{jn} + 2K_i^p D_p K_{jn} + K_j^p D_n K_{pi} + K_n^p D_j K_{pi}) \\ & - \epsilon^{ij}_m Y^p D_p D_i K_{jn} \\ & + \epsilon^{ij}_m (V_i (R_{jn} + K K_{jn}) - V_p K_j^p K_{ni} + K_n^p K_{pi} V_j) + \epsilon^{ij}_n G_{jm} V_i \\ & - \epsilon^{ij}_m (M_j^p D_i K_{np} + M_n^p D_i K_{jp} + M_i^p D_p K_{jn}). \end{aligned} \quad (14)$$

Here we've used the identity, valid in three dimensions,

$$R_{ijkl} = -\epsilon_{ij}^p \epsilon_{kl}^q G_{pq},$$

and G_{ij} is the Einstein tensor.

Note $C_{mn}^{(1)} g^{mn} = 0 = \epsilon^{mnr} C_{mn}^{(1)}$ identically, using both constraints, so that $C_{mn}^{(1)}$ is symmetric and trace-free and its vanishing can be expected to impose five linear conditions on Ψ_α .

- By considering $\epsilon^{ij}_m \epsilon^{pq}_n D_i D_j M_{pq}$ and using (7)–(10) we deduce the vanishing of

$$\begin{aligned} C_{mn}^{(2)} = & N \epsilon^{ij}_m \epsilon^{pq}_n (-2D_i D_p K_{qj} - 2K_{qj}(R_{ip} + K K_{ip})) - 2N(G_{in} K^i_m - K G_{mn}) \\ & + 2\epsilon^{ij}_m \epsilon^{pq}_n K_{qj} Y^k D_k K_{ip} - 2Y^i D_i G_{mn} \\ & - \epsilon^{ij}_m \epsilon^{pq}_n (2V_p D_i K_{qj} + 2V_i D_p K_{qj}) \\ & + 2\epsilon^{ij}_m \epsilon^{pq}_n K_{qj} (K_{is} M_p^s + K_{ps} M_i^s) + 2G_{mi} M_n^i + 2G_{ni} M_m^i, \quad (15) \end{aligned}$$

which is again symmetric and trace-free, so again is expected to give five equations.

The two constraints together amount to ten linear equations on the ten-dimensional space of Ψ_α so that there is an implied 10×10 -matrix A which we find more explicitly below. For non-trivial solutions to exist, this matrix must have zero determinant and the vanishing of the determinant corresponds to the vanishing of a polynomial in the tensors R_{ij} , $D_i R_{jk}$, K_{ij} , $D_i K_{jk}$ and $D_i D_j K_{km}$. This proves part 1 of Proposition 1.1. Similar conditions given in [3] involve higher orders in derivatives, specifically up to second order on R_{ij} and third order on K_{ij} .

From the way they are derived, these identities should be expected to be equivalent to the identity obtained as the vanishing of the Lie derivative of the Weyl tensor C_{abcd} along X :

$$0 = \mathcal{L}_X C_{abcd} = X^e \nabla_e C_{abcd} + C_{abce} \nabla_d X^e + C_{abed} \nabla_c X^e + C_{aecd} \nabla_b X^e + C_{ebcd} \nabla_a X^e,$$

and this in turn would give ten equations, linear in the variables Ψ_α .

To see that these two sets of identities are in fact equivalent we interpret the two $C_{mn}^{(i)}$ directly in terms of the Weyl tensor and its derivatives by following the $(3+1)$ -formalism given, for example, by Ellis and van Elst [5]. With n^a as the future-pointing, unit time-like normal to Σ , define electric and magnetic parts of the Weyl tensor by

$$E_{ab} = C_{acbd} n^c n^d, \quad B_{ab} = \frac{1}{2} \epsilon_{ac}^{ef} C_{efbd} n^c n^d.$$

Both are trace-free and symmetric tensors tangent to Σ and so may be written as E_{ij} and B_{ij} (but note that we have chosen the opposite sign on B_{ij} from [5]) and both are expressible in terms of the Cauchy data. These expressions are

$$E_{ij} = R_{ij} + K K_{ij} - K_{ik} K_j^k \quad (16)$$

$$B_{ij} = -\epsilon_i^{km} D_k K_{mj} \quad (17)$$

respectively.

We may express the Weyl tensor algebraically in terms of its electric and magnetic parts as

$$C_{abcd} = -4n_{[a} E_{b][c} n_{d]} - \epsilon_{ab}^{pq} \epsilon_{cd}^{rs} n_p E_{qr} n_s + 2\epsilon_{ab}^{pq} n_p B_{q[c} n_{d]} + 2\epsilon_{cd}^{pq} n_{[a} B_{b]p} n_q$$

so that the Weyl tensor, and indeed any tensor with Weyl tensor symmetries, is determined by its electric and magnetic parts. Note also that

$$C_{abcd} n^d = 2n_{[a} E_{b]c} - \epsilon_{ab}^{pq} n_p B_{qc}.$$

Now we use (16) and (17) in (14) and (15). The constraints become

$$\begin{aligned} C_{mn}^{(1)} &= N(\epsilon_{ij(m} D^i E_{n)}^j + \epsilon_m^{ij} \epsilon_n^{pq} K_{ip} B_{jq} + K_{i(m} B_{n)}^i - K B_{mn}) \\ &\quad + Y^k D_k B_{mn} + 2\epsilon_{ij(m} V^i E_{n)}^j + 2M_{(m}^k B_{n)k} = 0, \end{aligned}$$

and, removing a factor 2 for convenience,

$$\begin{aligned} \frac{1}{2} C_{mn}^{(2)} &= N(\epsilon_{ij(m} D^i B_{n)}^j - \epsilon_m^{ij} \epsilon_n^{pq} K_{ip} E_{jq} - K_{i(m} E_{n)}^i + K E_{mn}) \\ &\quad - Y^k D_k E_{mn} + 2\epsilon_{ij(m} V^i B_{n)}^j + 4E_{i(m} M_{n)}^i = 0. \end{aligned}$$

As a check, note the symmetry $E \rightarrow B \rightarrow -E$ relating $C^{(1)}$ and $C^{(2)}$.

We claim these equations express the vanishing of the electric and magnetic parts of the tensor Q_{abcd} defined by

$$\begin{aligned} Q_{abcd} &:= \mathcal{L}_X C_{abcd} \\ &= X^e \nabla_e C_{abcd} + C_{abce} \nabla_d X^e + C_{abed} \nabla_c X^e + C_{aecd} \nabla_b X^e + C_{ebcd} \nabla_a X^e. \end{aligned}$$

This is clearly a tensor with Weyl tensor symmetries and so is characterised in turn by its electric and magnetic parts. It is a straightforward matter to put Q_{abcd} into the $(3+1)$ -formalism and justify the claim. The two constraints are therefore equivalent to the vanishing of $\mathcal{L}_X C_{abcd}$ expressed in terms of the data. This proves the first half of part 2 of the Proposition. The rest is simpler in the two-component spinor formalism, which we turn to next.

3 Spinor form of the constraints

We have moved emphasis slightly from consideration of initial data to the question of when does a given 4-dim vacuum solution admit a Killing vector.³ This question is simpler to answer in 2-component spinors when the necessary condition is the existence of a nontrivial solution to (19) below, regarded as an equation in (X^a, ϕ_{AB}) , but there is an obvious necessary condition that is easy to obtain. The Weyl spinor has two complex invariants I, J given in (20) below, and it's clear that these must be constant along any Killing vector. Thus there can be no Killing vectors if these invariants define four functionally-independent real scalars, that is if

$$dI \wedge dJ \wedge d\bar{I} \wedge d\bar{J} \neq 0. \quad (18)$$

This will turn out to be equivalent to the vanishing of $C^{(1)}$ and $C^{(2)}$ as in the previous section, completing the proof of Proposition 1.1.

In the 2-component spinor formalism, the definition of the Lie derivative applied to the Weyl spinor ψ_{ABCD} leads to

$$\mathcal{L}_X \psi_{ABCD} := X^{EE'} \nabla_{EE'} \psi_{ABCD} + 4\phi_{(A}^E \psi_{BCD)E} = 0 \quad (19)$$

where ϕ_{AB} is a symmetric spinor obtained from the derivative of the Killing vector via

$$\nabla_{AA'} X_{BB'} = \phi_{AB} \epsilon_{A'B'} + \bar{\phi}_{A'B'} \epsilon_{AB}.$$

The system (19) consists of five complex equations for the four real components of X and the three complex components of ϕ_{AB} . Generically the Weyl spinor is invertible in the sense that there exists a symmetric spinor χ_{ABCD} with

$$\chi_{AB}{}^{CD} \psi_{CD}{}^{EF} = \delta_{(A}^{(E} \delta_{B)}^{F)}.$$

In fact this fails iff $J = 0$ as is seen as follows: recall the definition of the complex scalar invariants

$$I = \psi_{ABCD} \psi^{ABCD}, \quad J = \psi_{ABCD} \psi^{CDEF} \psi_{EF}{}^{AB}, \quad (20)$$

then the Cayley-Hamilton Theorem for the Weyl spinor thought of as a 3×3 matrix gives

$$\psi_{AB}{}^{EF} \psi_{EF}{}^{GH} \psi_{GH}{}^{PQ} - \frac{1}{2} I \psi_{AB}{}^{PQ} - \frac{1}{3} J \delta_{(A}^{(P} \delta_{B)}^{Q)} = 0$$

³This problem was addressed in [9] and the solution is in principle given there, in Theorem 6.1, but not in a very accessible form.

so that

$$\chi_{AB}{}^{PQ} = \frac{3}{J}(\psi_{AB}{}^{EF}\psi_{EF}{}^{PQ} - \frac{1}{2}I\delta_{(A}{}^{(P}\delta_{B)}{}^{Q)}),$$

which is evidently well-defined where $J \neq 0$.

We introduce the spinor $\chi_{E'ABCDE}$ by

$$\chi_{E'ABCDE} = \nabla_{E'E}\psi_{ABCD},$$

which is then symmetric in the unprimed indices by virtue of the vacuum Bianchi identity. In the NP formalism, by introducing a normalised spinor dyad, (19) can be written out in components as

$$\begin{pmatrix} \chi_{0'0} & \chi_{1'0} & \chi_{0'1} & \chi_{1'1} & 4\psi_1 & -4\psi_0 & 0 \\ \chi_{0'1} & \chi_{1'1} & \chi_{0'2} & \chi_{1'2} & 3\psi_2 & -2\psi_1 & -\psi_0 \\ \chi_{0'2} & \chi_{1'2} & \chi_{0'3} & \chi_{1'3} & 2\psi_3 & 0 & -2\psi_1 \\ \chi_{0'3} & \chi_{1'3} & \chi_{0'4} & \chi_{1'4} & \psi_4 & 2\psi_3 & -3\psi_2 \\ \chi_{0'4} & \chi_{1'4} & \chi_{0'5} & \chi_{1'4} & 0 & 4\psi_4 & -4\psi_3 \end{pmatrix} \begin{pmatrix} X^{00'} \\ X^{01'} \\ X^{10'} \\ X^{11'} \\ \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} = 0, \quad (21)$$

where $\chi_{A'a}$, ψ_a , ϕ_a are respectively the spinor components of $\chi_{E'ABCDE}$, ψ_{ABCD} and ϕ_{AB} , following NP conventions. In a block-matrix form (21) can be written:

$$\begin{pmatrix} M & N \end{pmatrix} \begin{pmatrix} X \\ \phi \end{pmatrix} = 0. \quad (22)$$

These are complex equations. Including the complex conjugates in the system gives the square system

$$\begin{pmatrix} M & N & 0 \\ \overline{M} & 0 & \overline{N} \end{pmatrix} \begin{pmatrix} X \\ \phi \\ \overline{\phi} \end{pmatrix} = 0, \quad (23)$$

so introduce the 10×10 matrix A (this matrix A is conjugate to the previous A) by

$$A = \begin{pmatrix} M & N & 0 \\ \overline{M} & 0 & \overline{N} \end{pmatrix}.$$

Then the necessary condition for existence of a nontrivial solution to (19) is the vanishing of $\det A$ and we turn next to this condition.

We first deal with the case when the Weyl spinor is type N. In this case we may choose the spinor dyad so that $\psi_0 \neq 0$ but $\psi_i = 0$ otherwise.

Then the matrix N has one column of zeroes so that the rank of A has dropped by at least two. In particular the determinant of A is zero and (23) has nontrivial solutions. This is a particular instance of the problem with algebraically-special metrics noted above.

If the Weyl spinor is not of type N then we may choose the spinor dyad (o^A, ι^A) so that $\psi_0 = 0 = \psi_4$, equivalently o^A and ι^A are independent principal spinors of the Weyl spinor. Now we can row-reduce the block matrix $(M \ N)$ to

$$\begin{pmatrix} * & 4\psi_1 & 0 & 0 \\ * & 0 & -2\psi_1 & 0 \\ * & 0 & 0 & -2\psi_1 \\ \tilde{r}_4 & 0 & 0 & 0 \\ \tilde{r}_5 & 0 & 0 & 0 \end{pmatrix}$$

where \tilde{r}_4, \tilde{r}_5 are combinations of the rows r_1, \dots, r_5 of M which we'll give below. Now

$$\det A = 256 \det \begin{pmatrix} \psi_1 \tilde{r}_4 \\ \psi_1^2 \tilde{r}_5 \\ \bar{\psi}_1 \tilde{r}_4 \\ (\bar{\psi}_1)^2 \tilde{r}_5 \end{pmatrix}$$

with

$$\psi_1 \tilde{r}_4 = \psi_1 r_4 - \frac{3}{2} \psi_2 r_3 + \psi_3 r_2 \quad (24)$$

and

$$\psi_1^2 \tilde{r}_5 = \psi_1^2 r_5 - 2\psi_1 \psi_3 r_3 + \psi_3^2 r_1. \quad (25)$$

Our assumptions on the spinor dyad imply that the Weyl spinor can be written

$$\psi_{ABCD} = -4\psi_3 o_{(A} o_B o_C \iota_{D)} + 6\psi_2 o_{(A} o_B \iota_C \iota_{D)} - 4\psi_1 o_{(A} \iota_B \iota_C \iota_{D)}$$

from which we may calculate the gradients of the two scalar invariants:

$$\nabla_{AA'} I = 2\psi^{BCDE} \nabla_{AA'} \psi_{BCDE} = 2\psi^{BCDE} \chi_{A'ABCDE}$$

and

$$\nabla_{AA'} J = 3\psi^{BCPQ} \psi^{DE}_{PQ} \chi_{A'ABCDE}.$$

Each can each be written as a pair of spinors by taking components on the index A . For I we obtain

$$\nabla_{0A'} I = 4(-2\psi_3 \chi_{A'1} + 3\psi_2 \chi_{A'2} - 2\psi_1 \chi_{A'3}), \quad \nabla_{1A'} I = 4(-2\psi_3 \chi_{A'2} + 3\psi_2 \chi_{A'3} - 2\psi_1 \chi_{A'4}) \quad (26)$$

while for J it is simpler to write the combination

$$\nabla_{0A'}J + \frac{3}{2}\psi_2\nabla_{0A'}I = 6(-\psi_3^2\chi_{A'0} + 2\psi_1\psi_3\chi_{A'2} - \psi_1^2\chi_{A'4}) \quad (27)$$

and

$$\nabla_{1A'}J + \frac{3}{2}\psi_2\nabla_{1A'}I = 6(-\psi_3^2\chi_{A'1} + 2\psi_1\psi_3\chi_{A'3} - \psi_1^2\chi_{A'5}). \quad (28)$$

Now comparing (24)–(25) with (26)–(28) and using (21) we see that $\det A$ is a (nonzero) constant multiple of

$$\det \begin{pmatrix} dI \\ dJ \\ d\bar{I} \\ d\bar{J} \end{pmatrix},$$

where each row is written as a 4-component vector, so that the non-vanishing of $\det A$ is indeed equivalent to (6). This completes the proof of Proposition 1.1.

As remarked above, in any algebraically-special space-time the Weyl spinor satisfies $I^3 = 6J^2$ so that (6) will not hold, and (19) admits solutions. More differentiation is needed to see whether these candidate KIDs are actual KIDs, but algebraically-special vacuum solutions without symmetries are known, for example some of the Robinson-Trautman solutions (see e.g. [6]). It remains a possibility that no algebraically-special metric without a symmetry can be asymptotically-flat⁴.

Another way to see this last part of the Proposition, in the particular case that $J \neq 0$, is to contract (19) with χ^{BCDF} . We obtain an expression for ϕ , namely

$$\phi_A^E = \frac{1}{3}\chi^{BCDE}X^{E'F}\chi_{E'AB C D F}. \quad (29)$$

Now contract (19) with ψ^{ABCD} and $\psi^{ABPQ}\psi_{PQ}{}^{CD}$ respectively to obtain

$$X^a\nabla_a I = 0 = X^a\nabla_a J. \quad (30)$$

Since the system (29)–(30) manifestly consists of five independent equations, it must exhaust (19), so that (19) is equivalent to defining ϕ_{AB} as in (29) and the two conditions (30) on X^a . Now it's clear that inconsistency of (19) must be (6).

⁴In this connection see [4].

4 Rank of the Linear System

We wish to confirm that the rank of the system in (23) is generically ten i.e. that generically $\det A \neq 0$ or equivalently (18) holds. To this end, we follow [3] and first claim that given any point p there is a metric such that any value of the Riemann tensor and its derivative, consistent with symmetries, holds at p . Consider the metric

$$g_{ab} = \eta_{ab} - \frac{1}{3}t_{acbd}x^cx^d + \frac{1}{6}t_{eacbd}x^ex^cx^d,$$

in pseudo-Cartesian (or inertial) coordinates x^a , where t_{acbd} and t_{eacbd} are constant tensors with all the symmetries respectively of the Riemann tensor R_{acbd} and its derivative $\nabla_e R_{acbd}$, then at the origin of coordinates one readily calculates that

$$R_{abcd} = t_{abcd}, \quad \nabla_e R_{abcd} = t_{eabcd}.$$

Thus the Riemann tensor and its derivative at a point can take any value allowed by symmetry.

We restrict to vacuum, so that the Ricci spinor and Ricci scalar are zero, and at a point p we make the choices

$$\psi_0 = \psi_2 = \psi_4 = 0, \quad \psi_1\psi_3 \neq 0, \quad |\psi_3| \neq 2|\psi_1|$$

for the Weyl spinor and

$$\chi_{A'0} = \chi_{A'3} = \chi_{A'4} = \chi_{A'5} = 0, \quad \chi_{A'1} = \iota_{A'}, \quad \chi_{A'2} = -i\sigma_{A'}$$

for its derivative. Then we may calculate

$$\nabla_a I = 8\psi_3(i\ell_a + n_a), \quad \nabla_a J = 6\psi_3(2i\psi_1\overline{m}_a - \psi_3m_a),$$

where $(\ell, n, m, \overline{m})$ form an NP-tetrad related in the conventional way to the normalised spinor dyad (o^A, ι^A) and its complex conjugate. Then

$$dI \wedge d\overline{I} \wedge dJ \wedge d\overline{J} = 4608i|\psi_3|^4(|\psi_3|^2 - 4|\psi_1|^2)\ell \wedge n \wedge m \wedge \overline{m} \neq 0.$$

This shows that (6) is not constrained to fail by symmetry alone, so that we can expect it to hold generically.

Appendix

The methods used here are also helpful for another problem considered in [3], namely the question of existence of a conformal Killing vector on a Riemannian 3-manifold (Σ, g) . We are led to

Proposition A.1: *Given a Riemannian 3-manifold (Σ, g) , with Cotton-York tensor Y_{ij} defined as in (35), and the tensor Z_{ij} as in (39) then this metric admits no conformal Killing vectors if the three conformally invariant scalars $\tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4$ defined below as polynomial invariants of Y_{ij} and Z_{ij} are functionally independent:*

$$d\tilde{\sigma}_2 \wedge d\tilde{\sigma}_3 \wedge d\tilde{\sigma}_4 \neq 0.$$

Thus the sufficient condition for the nonexistence of conformal Killing vectors, (42) below, is analogous to the condition (6) found above.

The existence of conformal Killing vectors can only depend on the conformal class of g and we should therefore expect conformal invariants and covariants to appear in the results. Recall the notion of conformal weight: under conformal rescaling

$$g_{ij} \rightarrow \hat{g}_{ij} = \Omega^2 g_{ij}$$

we say that a geometrical quantity η , a tensor or a scalar, has conformal weight w if

$$\eta \rightarrow \hat{\eta} = \Omega^w \eta.$$

A conformal Killing vector is a vector field X^i such that

$$2D_{(i}X_{j)} := \mathcal{L}_X g_{ij} = \frac{2}{3}\phi g_{ij}$$

for some function ϕ , which is evidently related to X^i by $\phi = D_i X^i$. Note that ϕ transforms inhomogeneously under conformal rescaling, according to

$$\phi \rightarrow \hat{\phi} = \phi + 3\mathcal{L}_X(\log \Omega).$$

We prolong as before, introducing a bivector F_{ij} so that

$$D_i X_j = \frac{1}{3}\phi g_{ij} + F_{ij}. \quad (31)$$

Then introduce the vector V_i by

$$D_i \phi = V_i, \quad (32)$$

and commute derivatives on (31) to obtain

$$D_i F_{jk} = -R_{ijk}{}^\ell X_\ell + \frac{1}{3}(V_j g_{ki} - V_k g_{ji}). \quad (33)$$

Now commute derivatives on this to obtain

$$D_i V_j = -3X^p D_p P_{ij} - 2\phi P_{ij} - 3P_{ip} F_j{}^p - 3P_{kj} F_i{}^p, \quad (34)$$

where P_{ij} is the Rho-tensor or Schouten tensor, defined in terms of the Ricci tensor and Ricci scalar by

$$P_{ij} := R_{ij} - \frac{1}{4}Rg_{ij},$$

(and called L_{ij} by [3]). Recall that in dimension 3 the Riemann tensor is given in terms of the Rho-tensor by

$$R_{ijkl} = P_{ik}g_{jl} + P_{jl}g_{ik} - P_{jk}g_{il} - P_{il}g_{jk}.$$

Following [3] we define the Cotton-York tensor by

$$Y_{ij} = \epsilon_i{}^{km} D_m P_{kj}, \quad (35)$$

(called H_{ij} by [3], also some authors have the opposite sign in this definition). This tensor is trace-free, symmetric, divergence-free and has conformal weight -1 , so that

$$Y_{ij} \rightarrow \hat{Y}_{ij} = \Omega^{-1} Y_{ij}$$

under conformal rescaling. It follows by an appeal to naturality that, for the conformal Killing vector X^i ,

$$\mathcal{L}_X Y_{ij} = -\frac{1}{3}\phi Y_{ij}, \quad (36)$$

but we shall derive this equation below.

The process of prolongation has led to a connection defined by (31)–(34) on the vector bundle $\Lambda^0 \oplus \Lambda^1 \oplus \Lambda^1 \oplus \Lambda^2$ with typical section $\psi_\alpha = (\phi, X_i, V_i, F_{ij})^T$. To construct the system we have already commuted derivatives on the first three of these so that the curvature of this connection is determined by commuting derivatives on the fourth. The result of doing this is then found to be precisely (36), which can be written at length as

$$X^p D_p Y_{ij} + F_i{}^p Y_{jp} + F_j{}^p Y_{ip} + \phi Y_{ij} = 0. \quad (37)$$

This is the first set of necessary conditions on ψ_α . The tensor on the LHS is symmetric and trace-free so that this represents five linear conditions on the ten components of ψ_α . We need more, but first we make two deductions from (37), by contracting with Y^{ij} and $Y^{ik}Y^j_k$ respectively. The terms containing F_{ij} drop out and we are left with

$$X^p D_p(Y_{ij}Y^{ij}) = -2\phi(Y_{ij}Y^{ij}), \quad X^p D_p(Y_{ij}Y^{jk}Y^i_k) = -3\phi(Y_{ij}Y^{jk}Y^i_k). \quad (38)$$

Note that $\sigma_1 := Y_{ij}Y^{ij}$ and $\sigma_2 := Y_{ij}Y^{jk}Y^i_k$ are conformally weighted scalars with weights -6 and -9 respectively so that their combination $\tilde{\sigma}_2 := \sigma_2/(\sigma_1)^{3/2}$ is a conformally-invariant scalar which by (38) is constant along X^i . (If $\sigma_1 = 0$ then Σ is conformally flat and everything about conformal Killing vectors is known.)

Next we must differentiate (37) to obtain more constraints. The divergence, or contracted derivative, automatically vanishes but the curl gives new information. First define the tensor

$$Z_{ij} = \epsilon_i^{mn} D_m Y_{nj}. \quad (39)$$

This is another trace-free, symmetric tensor but it is not conformally-weighted. In fact under conformal rescaling

$$Z_{ij} \rightarrow \hat{Z}_{ij} = \Omega^{-3}(Z_{ij} - 2\epsilon_{(i}^{mn} Y_{j)n} \Upsilon_m) \quad (40)$$

with $\Upsilon_i = D_i(\log \Omega)$ as usual.

Now the curl of (37) is

$$\begin{aligned} & \epsilon^{ikm} D_i (X^p D_p P_{kj} + F_k^p P_{jp} + F_j^p P_{kp} + \phi P_{kj}) \\ &= X^p D_p Z_j^m + \frac{4}{3} \phi Z_j^m + F_j^p Z_p^m - F_p^m Z_j^p + \frac{2}{3} \epsilon^{ikm} V_i Y_{kj} - \frac{1}{3} \epsilon_j^{km} Y_{kp} V^p. \end{aligned}$$

Relabelling indices and algebraically manipulating the last two terms gives the second necessary condition:

$$X^p D_p Z_{ij} + \frac{4}{2} Z_{ij} + F_i^k Z_{jk} + F_j^k Z_{ik} + \frac{2}{3} \epsilon_{pq(i} V^p Y_{j)}^q = 0. \quad (41)$$

The conditions (37) and (41) together give ten conditions on the ten components of ψ_α so that we can expect an obstruction to the existence of a conformal Killing vector. To see what it is, introduce the scalars

$$\sigma_3 = Z^{ij} Y_{ij}, \quad \sigma_4 = Z^{ij} Y_{ik} Y_j^k.$$

By (40) these are conformally-weighted scalars with weights -7 and -10 respectively, and by (37) and (41) we may calculate

$$X^p D_p(\log \sigma_3) = -\frac{7}{3}\phi, \quad X^p D_p(\log \sigma_4) = -\frac{10}{3}\phi.$$

We conclude that $\tilde{\sigma}_3 := \sigma_3/(\sigma_1)^{7/6}$ and $\tilde{\sigma}_4 := \sigma_4/(\sigma_1)^{5/3}$ are conformally-invariant scalars which are constant along X^i . Thus there can be no conformal Killing vector if the three conformally-invariant scalars that we have defined are functionally independent, that is if

$$d\tilde{\sigma}_2 \wedge d\tilde{\sigma}_3 \wedge d\tilde{\sigma}_4 \neq 0, \tag{42}$$

which is our necessary condition.

The corresponding condition in [3] is a polynomial in the tensors Y_{ij} , $D_i Y_{jk}$ and $D_i D_j Y_{km}$, which is the same order of differentials as (42).

References

- [1] R Bartnik and P Tod, A note on static metrics, *Class.Quant.Grav.* **23** (2006) 569-572 arXiv:gr-qc/0512097.
- [2] R Beig and P T Chruściel, Killing Initial Data, *Class.Quant.Grav.* **14** (1997) A83-A92 arXiv:gr-qc/9604040.
- [3] R Beig, P T Chruściel and R Schoen, KIDs are generic, *Annales Henri Poincaré* **6** (2005) 155-194 arXiv:gr-qc/0403042.
- [4] P T Chruściel and P Tod, On Mason's rigidity theorem. *Comm. Math. Phys.* **285** (2009), 1-29.
- [5] G F R Ellis and H van Elst, Cosmological models (Cargse lectures 1998) arXiv:gr-qc/9812046.
- [6] H Stephani, D Kramer, M MacCallum, C Hoenselaers and E Herlt, Exact solutions of Einstein's field equations. Second edition. Cambridge Monographs on Mathematical Physics Cambridge University Press, Cambridge, (2003).
- [7] T Branson, A Čap, M Eastwood, and R Gover, Prolongations of geometric overdetermined systems. *Internat. J. Math.* **17** (2006), 641664.
- [8] K P Tod, The integral constraint vectors of Traschen and three-surface twistors. *Gen. Relativity Gravitation* **20** (1988), 129-1308.

- [9] K Yano, The Theory of Lie Derivatives and its Applications, North Holland Pub Co Amsterdam (1957).